Critical Revision of de Haan's Tables of Definite Integrals.

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§ 1.

In 1858, D. Bierens de Haan published as "Vierde Deel" of the *Verhandelingen der Koninklijke Akademie van Wetenschappen* (Amsterdam) his "Tables d'Intégrales Définies." In this volume there are 7300 formulas, and these are accompanied by complete bibliographical references. De Haan then undertook the revision of these formulas, and the consideration of the underlying theory of Definite Integrals. Accordingly, in 1862, his "Exposé de la Théorie des Propriétés, des Formules de Transformation, et des Méthodes d'Évaluation des Intégrales Définies" was issued as Volume VIII in the same *Verhandelingen.* In this is included all his critical work of the intervening four years.

"Nouvelles Tables d'Intégrales Définies par D. Bierens de Haan" were published in 1867 (Leide). These tables contain 8339 formulas, of which 4200 come from the original tables, 2620 from the "Exposé" and the remainder from notes published at various times.

In this paper it is proposed to examine from the modern rigorous standpoint certain evaluations in the "Nouvelles Tables," and the theorems of the "Exposé" on which they depend. In nearly every case the formulas here considered will involve the principal value integral.

§ 2. *Definitions.*

De Haan first considers the function $f(x)$ defined over the interval $(a < b)$* and possessing the continuous primitive function $F(x)$. He employs Cauchy's† definition for the integral of a continuous function:

$$\int_a^b f(x)\,dx = \lim_{\delta=0} \sum_{\iota=1}^{n} (x_\iota - x_{\iota-1}) f(x_{\iota-1}),$$

* Here and elsewhere, unless the contrary is explicitly stated, all letters are finite.

† Cauchy, "Œuv. Comp.," Sér. 2, Tome 4, p. 122.

and gives also the definition $F(b) - F(a)$, which is dismissed as being equivalent to the other for the functions under consideration. The assumption that $f(x)$ is integrable in $(a < b)$ is not explicitly stated; this is important, for the function defined by Volterra* shows that $f(x)$ may satisfy the conditions here given and yet not be integrable.

Apparently de Haan means to include here not only proper integrals, but also certain improper integrals with infinite integrands; viz., those whose integrands have continuous primitives. However, Cauchy's definition will not apply to unlimited functions, for as δ approaches zero, the sum Σ can always be made to approach infinity.

Improper integrals with infinite fields are not separately defined in the "Exposé."

Improper integrals with infinite integrands: we shall compare the modern definition with that of de Haan, first of all.

1°. *De Haan's Definition.*

Let $f(x)$ possess the primitive function $F(x)$, in $(a < b)$.

Let $F(x)$ be continuous in $(a < b)$ except for an infinite discontinuity at c, $a < c < b$. Then we define

$$\int_a^b f(x)\,dx = \lim_{\theta=0} \left\{ \int_a^{c-\mu\theta} f(x)\,dx + \int_{c+\nu\theta}^b f(x)\,dx \right\}.$$

It is not explicitly stated, but from the context it is clear that μ, ν are any positive constants. It is also implied that for each choice of μ and ν the limit must exist, and have the same value for all such choices. This is the "general value" of the integral; when $\mu = \nu = 1$, we have the definition of the "principal value" (Cauchy).

2°. *The Modern Definition for this Case — One Singularity.*

Let $f(x)$ be regular† in $(a < b)$ except at c. Then we define

$$\int_a^b f(x)\,dx = \lim_{\theta=0} \int_a^{c-\theta} f(x)\,dx + \lim_{\theta=0} \int_{c+\theta}^b f(x)\,dx, \text{ if these exist;}$$

or

$$= \lim_{\theta_1=0,\, \theta_2=0} \left\{ \int_a^{c-\theta_1} f(x)\,dx + \int_{c+\theta_2}^b f(x)\,dx \right\}, \text{ if it exists.}$$

* *Giorn. di Battaglini*, Vol. XIX (1881).

† Lect. 1, p. 400, § 572.

3°. *The Principal Value Definition is Adopted by de Haan.*

There is no good reason, it is stated by de Haan,* for taking μ and ν other than unity. Accordingly it is decided to use the principal value integral throughout. This statement and decision are based upon the consideration of a single example, viz., $\int_0^\infty \frac{1}{1-x^2}\,dx$. His course of reasoning is as follows:

The primitive function of $\frac{1}{1-x^2}$, $F(x)=\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| = \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|$.

Now,

$$\begin{aligned}\int_0^\infty \frac{1}{1-x^2}\,dx &= \int_0^1 \frac{1}{1-x^2}\,dx + \int_1^\infty \frac{1}{1-x^2}\,dx \\ &= \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\Bigg]_0^1 + \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|\Bigg]_1^\infty \\ &= \frac{1}{2}\log\frac{2}{0} - 0 + 0 - \frac{1}{2}\log\frac{2}{0} \\ &= 0,\end{aligned}$$

which is the integral's true value.

The "general value" is

$$\lim_{\theta=0}\left\{\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\Bigg]_0^{1-\mu\theta} + \frac{1}{2}\log\left|\frac{1+1/x}{1-1/x}\right|\Bigg]_{1+\nu\theta}^\infty\right\} = \frac{1}{2}\log\frac{\nu}{\mu}.$$

The principal value, therefore, $=0$, which is the "true" value obtained above. But, $\frac{1}{2}\log\frac{2}{0}$ is *not a number,* and it is not possible to state that $\frac{1}{2}\log\frac{2}{0} - \frac{1}{2}\log\frac{2}{0} = 0$. Nor will it do to make so important an inference from the investigation of a *single* example.

Accordingly we may expect to find in the "Exposé" and in the "Nouvelles Tables" many integrals that exist as principal values only. In this paper these will be clearly indicated by the use of the symbol $P\int$, with suitable subscripts in each case.

4°. *Finite Number of Singularities: Cauchy's Definition of the Principal Value.*†

Let $f(x)$ become infinite at $c_1, c_2, \ldots, c_s$; $a < c_\iota < b$, $\iota = 1, 2, \ldots, s$. Then the principal value integrals are defined thus:

$$\int_a^b f(x)\,dx = \lim_{\theta=0}\left\{\int_a^{c_1-\theta} + \int_{c_1+\theta}^{c_2-\theta} + \cdots + \int_{c_s+\theta}^{b}\right\},$$

and

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\theta=0}\left\{\int_{-1/\theta}^{c_1-\theta} + \int_{c_1+\theta}^{c_2-\theta} + \cdots + \int_{c_s+\theta}^{1/\theta}\right\}.$$

* "Exposé," p. 7. † "Œuv. Comp.", Sér. 2, Tome 4, p. 143 — Résumé des leçons calc. inf. (1823).

According to these definitions, $f(x)$ may possess a principal value integral in $(a < b)$, and yet *not* possess a principal value integral in $(a < x_0)$, $x_0 \neq c_\iota$; or in $(x_1 < x_2)$, $a < x_1$, $x_2 < b$ and $x_1, x_2 \neq c_\iota$. Similarly in the case of the integral for $(-\infty, \infty)$.

5°. *Finite Number of Singularities: G. H. Hardy's Definition.**

Let $f(x)$ become infinite at $c_1, c_2, \ldots, c_s$; $a < c_\iota < b$, $\iota = 1, 2, \ldots, s$. Then Hardy defines

$$P\int_a^b f(x)\,dx = \operatorname*{limit}_{\theta_1=0,\ \theta_2=0,\ \ldots,\ \theta_s=0} \left\{ \int_a^{c_1-\theta_1} + \int_{c_1+\theta_1}^{c_2-\theta_2} + \cdots + \int_{c_s+\theta_s}^{b} \right\}.$$

According to this definition, if $f(x)$ possesses a principal value integral in $(a < b)$, it also possesses a principal value integral in any $(x_1 < x_2)$; $a < x_1$, $x_2 < b$; $x_1, x_2 \neq c_\iota$.

We may add the remark that, consistently with the above, if $f(x)$ has the singularities $c_1, c_2, \ldots, c_s$, and we take the principal value integral $\int_{-\infty}^{\infty} f(x)\,dx$, its convergence implies the convergence of each of the following principal value integrals:

$$\left\{ \int_{-\infty}^{x_1} + \int_{x_2}^{\infty} \right\}, \text{ where } -\infty < x_1 < c_1 \text{ and } c_s < x_2 < \infty;$$

and

$$\int_{x_1}^{x_2}, \text{ where } x_1, x_2 \neq c_\iota, \text{ but include, between themselves, at least one of the } c_\iota.$$

This definition* appears in the first of Hardy's four important papers on "The Theory of Cauchy's Principal Values," in which he establishes under fairly broad conditions many useful theorems on Infinite Series, Continuity, Inversion of Integration, Differentiation under the Integral Sign, etc. Frequent references we here made to these papers.

6°. *The Various Definitions Required here.*

(a) $$P_c\int_a^b f(x)\,dx = \lim_{\theta=0} \left\{ \int_a^{c-\theta} + \int_{c+\theta}^{b} \right\}, \quad \text{where } a < c < b.$$

See § 7, (4), (5) and (6).

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* *Proc. Lond. Math. Soc.*, Vol. XXXIV (1901), p. 18.

(b) $$P_{\pm\infty}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\Theta=\infty}\int_{-\Theta}^{\Theta} f(x)\,dx.$$

See § 8, (5).

(c) $$P_{\pm c}\int_{-c}^{c} f(x)\,dx = \lim_{\theta=0}\int_{-c+\theta}^{c-\theta} f(x)\,dx.$$

See § 6, (3), (5), (7) and (9).

(d) $$P_{\pm c}\int_{a}^{b} f(x)\,dx = \lim_{\theta=0}\int_{-c+\theta}^{c-\theta} f(x)\,dx + \lim_{\theta=0}\left\{\int_{a}^{-c-\theta} + \int_{c+\theta}^{b}\right\},$$

where $a < -c < c < b$.

See § 6, (4); cf. (8).

(e) $$P_{\{c_\iota\}}\int_{a}^{b} f(x)\,dx = \lim_{\theta=0}\left\{\int_{a}^{c_1-\theta} + \int_{c_1+\theta}^{d_1}\right\} + \lim_{\theta=0}\left\{\int_{d_1}^{c_2-\theta} + \int_{c_2+\theta}^{d_2}\right\}$$

$$+ \ldots + \lim_{\theta=0}\left\{\int_{d_{s-1}}^{c_s-\theta} + \int_{c_s+\theta}^{b}\right\},$$

where $c_{\iota-1} < d_{\iota-1} < c_\iota$,

or $$P_{\{c_\iota\}} = \sum_{\iota=1}^{s} P_{c_\iota},$$ to put it briefly.

See § 4, (25) and (27); § 6, (8).

This last agrees with Hardy's definition given in 5° above. The meaning of such symbols as $P_{c,\,\pm\infty}$, $P_{-d,\,e,\,\pm f,\,\pm\infty}$, etc., is at once evident.

§ 3. *Abbreviations, etc.*

VIII: Verhandelingen der Koninklijke Akademie van Wetenschappen, Deel VIII, 1862. Exposé de la théorie, des proprietés, etc.

Nouv. Tab.: Nouvelles Tables d'Intégrales Définies, par D. B. de Haan, 1867.

T 172, F 4: Table 172, Formula 4, in the Nouvelles Tables.

Lect. I: Lectures on the Theory of Functions of Real Variables, Volume I, by James Pierpont, 1905.

si (u): $\int_0^u \frac{\sin x}{x}dx$, u in the domain $(-\infty, \infty)$.

§ 4.

$$\int_0^\infty \frac{\sin px\,dx}{(q^2-x^2)\,.\,x} = \frac{\pi}{2q^2}(1-\cos pq), \qquad \text{Nouv. Tab., T 172, F 4.} \tag{1}$$

We shall prove the following:

$$\left.\begin{aligned} P_q\int_0^\infty \frac{\sin px\,dx}{(q^2-x^2)\,.\,x} &= \frac{\pi}{2q^2}(1-\cos pq); \quad p \gtreqless 0,\ q>0, \\ \text{or} \qquad &= -\frac{\pi}{2q^2}(1-\cos pq); \quad p<0,\ q>0. \end{aligned}\right\} \tag{2}$$

At $x=0$, set the integrand, $f(x,p) = p/q^2$.

We shall show that we may differentiate twice under the integral sign. Consider the integral, I, in three parts:

$$I_1 = \int^a, \quad I_2 = P_q \int_a^b, \quad I_3 = \int_b^\infty; \quad \text{where } 0 < a < q < b < \infty.$$

1°.
$$\frac{d^2 I_1}{dp^2} = \int_0^a \frac{\partial^2 f}{\partial p^2}\, dx. \tag{3}$$

In $R_1 = (0 \leqq x \leqq a;\ 0 \leqq p \leqq p_1)$ both $f(x, p)$ and the derivative $f'_p(x, p)$ are continuous. Therefore,

$$\frac{dI_1}{dp} = \int_0^a \frac{\partial f}{\partial p}\, dx. \tag{4}$$

Again, in R_1, both $f'_p(x, p)$ and its derivative $f''_{pp}(x, p)$ are continuous Hence from (4) we get (3), by differentiating again.

2°.
$$\frac{d^2 I_2}{dp^2} = P_q \int_a^b \frac{\partial^2 f}{\partial p^2}\, dx. \tag{5}$$

$$R_2 = (a \leqq x \leqq b;\ 0 \leqq p \leqq p_1).$$

Let $f(x, p) = \dfrac{1}{q - x} \cdot \Theta(x)$, where $\Theta(x) = \dfrac{\sin px}{x(q + x)}$.

The derivatives of $\Theta(x)$ are

$$\Theta'_p = \frac{\cos px}{q + x} \quad \text{and} \quad \Theta'_x = \frac{px(q + x)\cos px - \sin px\,(q + 2x)}{x^2 (q + x)^2},$$

both of which are continuous in R_2. Moreover, q is a constant independent of p. A corollary of Hardy's* justifies differentiation under the principal value integral sign:

$$\frac{dI_2}{dp} = P_q \int_a^b \frac{\partial f}{\partial p}\, dx = P_q \int_a^b \frac{\cos px}{q^2 - x^2}\, dx. \tag{6}$$

We may apply this same course of reasoning again, having now:

$$f(x) = \frac{\cos px}{q^2 - x^2}, \qquad \Theta(x) = \frac{\cos px}{q + x},$$

$$\Theta'_p = \frac{-x \sin px}{q + x} \quad \text{and} \quad \Theta'_x = \frac{p(q + x)\sin px - \cos px}{(q + x)^2}.$$

We obtain (5).

3°.
$$\frac{d^2 I_3}{dp^2} = \int_b^\infty \frac{\partial^2 f}{\partial p^2}\, dx. \tag{7}$$

$$R''_3 = (b \leqq x < \infty,\quad 0 < p_0 \leqq p \leqq p_1) = (x \text{ in } \mathfrak{A}_3,\ p \text{ in } \mathfrak{B}''_3).\dagger$$

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* *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 84.

† v. Lect. I, p. 387.

$f(x, p)$ is regular in $R_3' = (b, \infty\,;\ 0, p_1)$ and continuous in p for each x in $\mathfrak{A}_3$, $f_x'(x, p)$ is regular in R_3' and uniformly continuous in p in any rectangle $(b, B;\ 0, p_1)$. Since $|f_p'| \leqq \frac{1}{q^2 - x^2}$, $\int_0^\infty f_p'$ is uniformly convergent in $\mathfrak{B}_3'$. Therefore, we have:*

$$\frac{dI_3}{dp} = \int_0^\infty \frac{\partial f}{\partial p}\, dx \text{ in } (0, p_1). \tag{8}$$

We may differentiate under the integral sign again, for p in $(0, p_1)$. In this case, the derivative $f_{pp}'' = \frac{-x \sin px}{q^2 - x^2}$ is integrable in (b, ∞) for each p in $\mathfrak{B}_3''$, since $\int_b^x \sin px$ is limited in $\mathfrak{A}_3$, $\frac{-x}{q^2 - x^2}$ is monotone in $V(\infty)$ and $\doteq 0$ as $x \doteq \infty$.† And $\int_b^\infty dx$ converges uniformly in $\mathfrak{B}_3'' = (p_0, p_1)$, since we have

$$\int_b^\infty dx = \int_b^\infty \frac{\sin px}{x} \cdot \frac{1}{1 - q^2/x^2}\, dx,$$

where $\int_b^\infty \frac{\sin px}{x}\, dx$ converges uniformly in $\mathfrak{B}_3''$ and $\frac{1}{1 - q^2/x^2}$ is monotone.‡

Hence§ we may write:

$$\frac{d^2 I_3}{dp^2} = \int_0^\infty \frac{\partial^2 f}{\partial p^2}\, dx \quad \text{in } (p_0, p_1). \tag{7}$$

4°. Bringing together the results of 1°, 2° and 3°:

$$\frac{d^2 I}{dp^2} = P_q \int_0^\infty \frac{\partial^2 f}{\partial p^2}\, dx \quad \text{in } (p_0, p_1), \tag{9}$$

also
$$\frac{dI}{dp} = P_q \int_0^\infty \frac{\partial f}{\partial p}\, dx \quad \text{in } (0, p_1). \tag{10}$$

(9) gives
$$\frac{d^2 I}{dp^2} = \int_0^\infty \frac{\sin px\, dx}{x} - q^2 P_q \int_0^\infty \frac{\sin px\, dx}{x(q^2 - x^2)} \tag{11}$$

$$= \frac{\pi}{2} - q^2 I \quad \text{if } p > 0, \tag{12}$$

or
$$= 0 \qquad\qquad \text{if } p = 0.$$

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* Lect. I, p. 493, § 683, 2. This theorem holds true if f_y' is uniformly continuous in y in any arbitrary interval $(a < b)$, except on the lines $x = a_1$. Cf. Lect. I, §§ 666, 669.

† Lect. I, p. 453, § 643.

‡ Lect. I, p. 470; p. 468, § 661, 2.

§ Lect. I, p. 493, § 683, 2.

Now, multiplying (12) by $2\dfrac{dI}{dp}$ and integrating, we have

$$\left(\frac{dI}{dp}\right)^2 = \pi I - 2q^2 I^2 + C.$$

Setting $p=0$ and using T 17, F 1 (see § 4):

$$C=0 \quad \text{and} \quad \frac{dI}{dp} = \pm\sqrt{\pi I - 2q^2 I}.$$

Integrating again, setting $p=0$, we have $I = \dfrac{\pi}{2q^2}(1-\cos pq)$; $p \gtreqless 0$. (2a)

Similarly, making the proper changes in (11) we get the result for $p<0$. (2b)

Criticism.

(a) For T 172, F 4 to be true, it is necessary that $p \gtreqless 0$. This is shown clearly at equation (12), § 4, where the first integral of (11) is $-\frac{\pi}{2}$ if $p<0$.

(b) The condition on q given in (2) is necessary. For if $q=0$, the integral is divergent in any interval $(0<a)$. If $q<0$ we must consider $P_{|q|}$.

(c) In accordance with de Haan's definition of the improper integral, the notation in T 172, F 4 conveys no hint of the fact that the integral exists only as a principal value.

(d) The method of proof used here is that of de Haan in VIII Méth. 25, N 3, p. 522; where, however, differentiation under the integral sign is employed twice without any justification.

$$\int_0^\infty \frac{\cos px\,dx}{q^2-x^2} = \frac{\pi}{2q}\sin pq, \qquad \text{Nouv. Tab., T 161, F 5.} \tag{13}$$

This becomes:

$$\left.\begin{aligned} P_q\int_0^\infty \frac{\cos px\,dx}{q^2-x^2} &= \frac{\pi}{2q}\sin pq;\quad p \gtreqless 0,\ q>0,\\ \text{or}\qquad &= -\frac{\pi}{2q}\sin pq;\quad p<0,\ q>0. \end{aligned}\right\} \tag{14}$$

According to the preceding proof, equation (2) may be differentiated under its integral sign, giving (14). See, again, VIII Méth. 25, N 3, p. 522. This integral is evaluated in one of Hardy's papers* also.

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* *Quart. Journ. P. A. Math.*, Vol. XXXII (1901): "On Differentiation and Integration under the Integral Sign."

$$\int_0^\infty \frac{x \sin px\, dx}{q^2 - x^2} = -\frac{\pi}{2} \cos pq, \qquad \text{Nouv. Tab., T 161, F 4} \tag{15}$$

(see "Corrections" also).

This becomes:

$$P_q \int_0^\infty \frac{x \sin px\, dx}{q^2 - x^2} = -\frac{\pi}{2} \cos pq; \quad p > 0,\ q > 0,$$

or

$$= \frac{\pi}{2} \cos pq; \quad p < 0,\ q > 0. \tag{16}$$

As before, by differentiating under the integral sign in (13), we get (15). See again the references given under (13).

$$\int_0^\infty \frac{\sin px \sin rx\, dx}{q^2 - x^2} = -\frac{\pi}{2q} \cos pq \,.\, \sin qr; \quad p > r,$$

or

$$= -\frac{\pi}{4q} \sin 2pq; \quad p = r,$$

or

$$= -\frac{\pi}{2q} \sin pq \,.\, \cos qr; \quad p < r, \tag{17}$$

Nouv. Tab., T 166, F 1.

These become:

$$P_q \int_0^\infty \frac{\sin px \sin rx}{q^2 - x^2}\, dx = -\frac{\pi}{2q} \cos pq \,.\, \sin qr; \quad p \overline{\gtrless} r \overline{\gtrless} 0,\ q > 0,$$

or

$$= -\frac{\pi}{2q} \sin pq \,.\, \cos qr; \quad r \overline{\gtrless} p \overline{\gtrless} 0,\ q > 0. \tag{18}$$

In the proof use (14) = T 161, F 5, writing

$$\sin px \,.\, \sin rx = \tfrac{1}{2} \cos (p - r) x - \tfrac{1}{2} \cos (p + r) x.$$

And similarly T 166, F 2 and F 3.

$$\int_0^\infty \frac{x \sin px\, dx}{(q^2 - x^2)^2} = -\frac{p\pi}{4q} \sin pq, \qquad \text{Nouv. Tab., T 171, F 1.} \tag{19}$$

$$\int_0^\infty \frac{x^3 \sin px\, dx}{(q^2 - x^2)^2} = \frac{\pi}{4} (2 \cos pq - pq \sin pq), \qquad \text{Nouv. Tab., T 171, F 2.} \tag{20}$$

$$\int_0^\infty \frac{\cos px\, dx}{(q^2 - x^2)^2} = \frac{\pi}{4q^3} (\sin pq - pq \cos pq), \qquad \text{Nouv. Tab., T 171, F 3.} \tag{21}$$

$$\int_0^\infty \frac{x^2 \cos px\, dx}{(q^2 - x^2)^2} = \frac{-\pi}{4q} (\sin pq + pq \cos pq), \qquad \text{Nouv. Tab., T 171, F 4.} \tag{22}$$

But these four do not even exist, if $q > 0$, either as integrals or as principal value integrals, except in the degenerate case of (20), where $p > 0$ and $q = 0$ give the integral used above at (12).

De Haan derives these in VIII at p. 565, where they all depend upon the differentiation under the integral sign of (14) = T 161, F 5 and (15) = T 161, F 4.

$$\int_0^\infty \frac{x \sin px\, dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2(q^2 - r^2)} (\cos pq - \cos pr), \tag{23}$$

Nouv. Tab., T 174, F 3.

$$\int_0^\infty \frac{x^3 \sin px\, dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2(q^2 - r^2)} (r^2 \cos pr - q^2 \cos pq), \tag{24}$$

Nouv. Tab., T 174, F 4.

The first of these becomes:

$$P_{q,r} \int_0^\infty \frac{x \sin px\, dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2(q^2 - r^2)} (\cos pq - \cos pr); \quad p > 0,\ q > r > 0. \tag{25}$$

And exactly the same revision is necessary in (24). (26)

In VIII, p. 331, de Haan decomposes these integrands into partial fractions and uses T 161, F 4 in both cases.

Similarly, to T 175, F 3 and F 4 the principal value sign and conditions on the constants must be added. Then we have:

$$P_{q,r} \int_0^\infty \frac{\cos px\, dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2qr(q^2 - r^2)} (q \sin pr - r \sin pq); \quad p \gtreqless 0,\ q > r > 0. \tag{27}$$

$$P_{q,r} \int_0^\infty \frac{x^2 \cos px\, dx}{(q^2 - x^2)(r^2 - x^2)} = \frac{\pi}{2(q^2 - r^2)} (r \sin pr - q \sin pq); \quad p \gtreqless 0,\ q > r > 0. \tag{28}$$

As in the last group, we use T 161, F 5, as at VIII, p. 331, where, however, the evaluations given in F (484) and F (485) are wrong.

$$\int_0^\infty \text{arc tan} \frac{x}{q} \cdot \frac{dx}{(p - x)^2} = \frac{1}{p^2 + q^2} \left(q \log \frac{q}{p} - \frac{1}{2} p\pi \right), \tag{29}$$

Nouv. Tab., T 249, F 2.

$$\int_0^\infty \text{arc tan} \frac{x}{q} \cdot \frac{x\, dx}{(p^2 - x^2)^2} = \frac{-\pi}{4(p^2 + q^2)}, \quad \text{Nouv. Tab., T 249, F 4.} \tag{30}$$

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{q}\cdot\frac{dx}{(p-x)^2}=\frac{q}{p(p^2+q^2)}\left(p\log\frac{p}{q}+\frac{1}{2}q\pi\right), \tag{31}$$

Nouv. Tab., T 249, F 9.

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{q}\cdot\frac{x\,dx}{(p^2-x^2)^2}=\frac{-\pi q^2}{4p^2(p^2+q^2)},\quad \text{Nouv. Tab., T 249, F 11.} \tag{32}$$

But, these four are not even convergent at $x=p$, $p>0$, since $f(x)$ has a pole of order 2 there, and $\frac{1}{(p-x)^2}>0$ on each side of that point.

De Haan derives these at VIII, p. 595, from VIII, p. 230, F (207) by the method of integration by parts.

$$\int_0^\infty \frac{dx}{q^2-x^2}=0,\qquad \text{Nouv. Tab., T 17, F 1.} \tag{33}$$

This becomes:

$$P_{|q|}\int_0^\infty \frac{dx}{q^2-x^2}=0;\quad q\neq 0.$$

$$\int_0^\infty \frac{dx}{(q^2-x^2)^2}=0 \ \text{(v. T 17, F 1)},\qquad \text{Nouv. Tab., T 17, F 17.} \tag{34}$$

This integral does not exist. De Haan derives it from T 17, F 1.

§ 5.

$$\int_a^b \operatorname{si}(rx)\,.\,\phi(x)\,dx=\int_0^1\frac{dy}{y}\int_a^b \sin rxy\,.\,\phi(x)\,.\,dx, \tag{1}$$

VIII, Méth. 18, N 24, p. 461, Théorème (XXXII).

We shall prove the following theorem:

Let $\Theta(x)$, $\Theta'(x)$ be continuous, and limited in $\mathfrak{A}=(0,\infty)$.

Let $\frac{\Theta(x)}{q-x}$ be absolutely integrable in (a,∞), where $0<q<a$.

Let
$$f(x,y)=\frac{\Theta(x)}{q-x}\cdot\frac{\sin rxy}{y},\quad \textit{for } y>0,\ x\neq q,$$
$$=\frac{\Theta(x)}{q-x}\cdot r,\qquad \textit{for } y=0,\ x\neq q.$$

Then
$$P_q\int_0^b dx\int_0^1 f(x,y)\,dy=\int_0^1 dy\,P_q\int_0^b f(x,y)\,dx; \tag{2}$$

b finite or infinite, $q<b$.

CASE 1°: b finite.

$P_q \int_0^b f\,dx$ is convergent for each y in $\mathfrak{B} = (0, 1)$;* Moreover it is uniformly convergent in $\mathfrak{B}$.† For (i) there are no other infinite discontinuities aside from $x = q$ and (ii) at the point $x = q$ we have:

$$\varepsilon > 0;\ \theta_0 > 0;\ \left| P_q \int_{q-\theta}^{q+\theta} \frac{\Theta(x) . \sin rxy . dx}{(q-x) . y} \right|,\quad \text{by a corollary of Hardy's,‡}$$

$$= \left[\frac{\partial}{\partial x} \frac{\Theta(x) . \sin r(x) y}{y} \right]_{x=q+\mu} . 2\theta;\quad -\theta \leqq \mu \leqq \theta,$$

$$= \left\{ \Theta'(q+\mu) \frac{\sin r(q+\mu) y}{y} + \Theta(q+\mu) . r \cos (q+\mu) y \right\} . 2\theta,$$

$$\leqq M . 2\theta;\quad M = \text{max. of } \{\ \} \text{ in } \mathfrak{A} = (0, b)$$

$$< \varepsilon \text{ for each } \theta < \theta_0 \text{ and all } y \text{ in } \mathfrak{B},$$

if we take $\theta_0 < \varepsilon / 2M$.

Now, by changing the value of f along $y = 0$ to $\dfrac{\Theta(x)}{q-x} . rx$, f becomes continuous in R except along $x = q$. Therefore,§

$$P_q \int_0^b dx \int_0^1 f\,dy = \int_0^1 dy\ P_q \int_0^b f\,dx;\quad b \text{ finite.}$$

CASE 2°: b infinite.

We must prove that $\displaystyle\int_a^\infty dx \int_0^1 f\,dy = \int_0^1 dy \int_a^\infty f\,dx$, where $q < a$.

$f(x, y)$ having no infinite discontinuities in $R = (a, \infty;\ 0, 1)$ and being integrable as to x and as to y in R, is regular.

$\displaystyle\int_a^\infty f\,dx$ converges uniformly in $\mathfrak{B}$, except at 0, since $|f| \leqq \left| \dfrac{\Theta(x)}{q-x} . \dfrac{1}{\eta} \right|$ for every y in $(\eta < 1)$; $\eta > 0$ and this is integrable in $\mathfrak{A}$.

$\displaystyle\int_0^1 f\,dy$ is a proper integral.

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 24.

† G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 66, Definition.

‡ Vol. XXXIV, pp. 26, 27.

§ G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 94.

Again, $\int_a^\infty dy \int_0^y f\,dy$ converges uniformly in $\mathfrak{B}$.* For, $\dfrac{\Theta(x)}{q-x}$ being continuous in $\mathfrak{A}$ and $\int_0^y \dfrac{\sin r\,xy\,dy}{y}$ being limited in R, $\left(0 \leqq \int_0^y \dfrac{\sin r\,xy}{y} \leqq \pi,\ r > 0\right)$, $\dfrac{\Theta(x)}{q-x}\int_0^y \dfrac{\sin r\,xy\,dy}{y}$ is regular in R; $\dfrac{\Theta(x)}{q-x}$ is absolutely integrable in $\mathfrak{A}$; and finally $\int_0^y \dfrac{\sin r\,xy\,dy}{y}$ is not only limited in R but also is a continuous function of x in $(a < G)$ for each y in $\mathfrak{B}$,† and therefore integrable in any $(a < G)$.

Hence‡ we have the theorem:

$$\int_a^\infty dx \int_0^1 f\,dy = \int_0^1 dy \int_a^\infty f\,dx.$$

1° and 2° together give the theorem:

$$P_q \int_0^b \frac{\Theta(x)\,.\,\mathrm{si}\,(rx)\,dx}{q-x} = \int_0^1 \frac{dy}{y}\, P_q \int_0^b \frac{\Theta(x)\sin r\,xy\,dx}{q-x}, \tag{2}$$

$q < b$, b finite or infinite.

$$\int_0^\infty \frac{\sin px\ \mathrm{si}\,(rx)\,dx}{q^2-x^2} = -\frac{\pi}{2q}\cos pq\,.\,\mathrm{si}\,(qr), \quad p \gtreqless r, \quad \text{Nouv. Tab., T 463, F 1,}$$

or

$$= -\frac{\pi}{2q}\cos pq\,.\,\mathrm{si}\,(pq) + \frac{\pi}{2q}\sin pq\,\{\mathrm{ci}\,(pq) - \mathrm{ci}\,(qr)\}, \quad p \leqq r. \tag{3}$$

This integral should be $P_q \int_0^\infty$ and the conditions should be

$$p \gtreqless r \gtreqless 0,\ q > 0 \quad \text{and} \quad r \gtreqless p \gtreqless 0,\ q > 0 \quad \text{respectively.} \tag{4}$$

For, apply the preceding theorem (2), letting $\Theta(x) = \dfrac{\sin px}{q+x}$.

Then $\Theta'(x) = \dfrac{p(q+x)\cos px - \sin px}{(q+x)^2}$ and $\Theta(x)$ are both continuous and limited in $\mathfrak{A} = (0, \infty)$.

$\dfrac{\sin px}{q^2-x^2}$ is absolutely integrable in (a, ∞); $a > q$.

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* Lect. I, p. 467, § 660, 1.

† Lect. I, p. 390, § 563, 3.

‡ Lect. I, p. 483, § 674, 1.

The conditions of the theorem being satisfied, we have:

$$P_q \int_0^\infty \frac{\sin px \,\mathrm{si}\,(rx)\, dx}{q^2 - x^2} = \int_0^1 \frac{dy}{y}\, P_q \int_0^\infty \frac{\sin rxy \sin px}{q^2 - x^2}\, dx$$
$$= -\frac{\pi}{2q} \cos pq \,.\, \mathrm{si}\, qr,$$

using § 4, (17) = T 166, F 1.

In the other case where $r \overline{\gtrless} p$, the interval of integration $\mathfrak{B} = (0, 1)$ must be divided into two parts; all else is similar.

Criticism.

De Haan gives Théorème (XXXII)* as an immediate application of his "Méthode 18." This method (replacing a factor by a definite integral) is due to Cauchy, being given in his "Mémoire sur diverses formules relatives à la théorie des intégrales définies, etc."† De Haan's statement‡ is as follows:

"Let $\int_p^q g(x, y)\, dy = h(x)$,

"Then $\int_a^b h(x) \,.\, \phi(x)\, dx = \int_p^q dy \int_a^b g(x, y) \,.\, \phi(x)\, dx$, provided that $g(x, y) \,.\, \phi(x)$ does not become discontinuous for any value of x in $(a < b)$ and of y in $(p < q)$."

This theorem will be useful in cases where we know the value of $\int_a^b g \,.\, \phi\, dx$.

The sufficient condition here given by de Haan is not found in Cauchy, and, indeed, is overlooked by de Haan himself in applying Théorème (XXXII) to T 463, F 1. For in that formula $\phi(x) = \frac{\sin px}{q^2 - x^2}$, which is not continuous in $R = (0, \infty;\ 0, 1)$.

Again, from this application, it is clear that the theorem is held to be applicable in case one of the integrals, $\int_a^b f\, dx$, becomes a principal value. From our standpoint, this must be explicitly stated. In the case of the principal value we must use such theorems as those of Hardy, employed in the proof above.

Finally, from this same context again, we see that b is meant to be infinite as well as finite. This contingency is covered by Case 2° above.

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* VIII, Méth. 18, N 24, p. 461.

† "Œuv. Comp.", Sér. 2, Tome 1, p. 467.

‡ VIII, p. 437.

§ 6.

$$\int_{-a}^{a} \frac{\sin x}{\cos 2x}\, dx = 0, \quad a < \pi/4, \qquad \text{VIII, p. 231, F(210).} \tag{1}$$

$$= 0, \quad a \gtreqless \pi/4, \qquad \text{VIII, p. 231, F(211).} \tag{2}$$

De Haan gives the primitive function

$$F(x) = \int \frac{\sin x}{\cos 2x}\, dx = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2}\, .\, \cos x}{1 - \sqrt{2}\, .\, \cos x}.$$

We note that the integrand, $f(x)$, is an odd function, $F(x)$ an even function.

When $a < \pi/4$, $\int_{-a}^{a} = 0$, since f is odd. (1)

When $a = \pi/4$, $P_{\pm\pi/4} \int_{-\pi/4}^{\pi/4} \frac{\sin x\, dx}{\cos 2x} = 0$, using $F(x)$. (3)

When $\pi/4 < a < 3\pi/4$, $P_{-\pi/4,\, \pi/4} \int_{-a}^{a} = 0$; also $P_{\pm\pi/4} \int_{-a}^{a} = 0$. (4)

When $a = 3\pi/4$, $P_{-\pi/4,\, \pi/4,\, \pm 3\pi/4} \int_{-3\pi/4}^{3\pi/4} = 0$. (5)

And so on, for $3\pi/4 < a \leqq \infty$, taking in each case the proper sort of principal value integral.

$$\int_{-a}^{a} \frac{\cos x\, dx}{\cos 2x} = \frac{1}{\sqrt{2}} \log \frac{1 + \sin \sqrt{2}\, a}{1 - \sin \sqrt{2}\, a}, \quad a < \pi/4, \qquad \text{VIII, p. 231, F(212).} \tag{6}$$

$$\int_{-a}^{a} \frac{\cos x\, dx}{\cos 2x} = \infty, \quad a \gtreqless \pi/4, \qquad \text{VIII, p. 231, F(213).} \tag{7}$$

F (213) is wrong except for the values $a = (2n + 1)\pi/4$.

De Haan gives the primitive function

$$F(x) = \int \frac{\cos x}{\cos 2x} = \frac{1}{2\sqrt{2}} \log \frac{1 + \sqrt{2} \sin x}{1 - \sqrt{2} \sin x}.$$

$f(x)$ is an even function.

When $a < \pi/4$, we get F (212) at once. (6)

When $a = \pi/4$, $P_{\pm\pi/4} \int_{-\pi/4}^{\pi/4} \frac{\cos x\, dx}{\cos 2x}$ is divergent. (7)

When $\pi/4 < a < 3\pi/4$, $P_{-\pi/4,\, \pi/4} \int_{-a}^{a} = P_{-\pi/4} \int_{-a}^{0} + P_{\pi/4} \int_{0}^{a}$

$$= \frac{1}{\sqrt{2}} \log \frac{1 + \sqrt{2} \sin a}{1 - \sqrt{2} \sin a}. \tag{8}$$

Comparing (4), we note that here $P_{\pm\pi/4}\int_{-a}^{a}$ does not converge.

When $a=3\pi/4$, $P_{-\pi/4,\ \pi/4,\ \pm 3\pi/4}\int_{-3\pi/4}^{3\pi/4}=+\infty$. (9)

When $3\pi/4<a<5\pi/4$, $P_{-3\pi/4,\ -\pi/4,\ \pi/4,\ 3\pi/4}\int_{-a}^{a}=$ etc. (10)

And so on as in the previous case, at (8).

Instead of F (213) we have

$$P\int_{-a}^{a}\frac{\cos x\,dx}{\cos 2x}=\frac{1}{\sqrt{2}}\log\frac{1+\sqrt{2}\sin a}{1-\sqrt{2}\sin a},\quad a>\pi/4\neq(2n+1)\pi/4, \tag{11}$$

and $P\int_{-a}^{a}\frac{\cos x\,dx}{\cos 2x}$ is divergent, $a=(2n+1)\pi/4$. (12)

§ 7.

$$\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x\,dx}{x^2-q^2}=\frac{\pi}{4}\log\frac{p^2+q^2}{q^2},\quad \text{Nouv. Tab., T 248, F 10.} \tag{1}$$

This becomes

$$P_q\int_0^\infty \operatorname{arc\,cot}\frac{x}{p}\cdot\frac{x\,dx}{x^2-q^2}=\frac{\pi}{4}\log\frac{p^2+q^2}{q^2};\quad p>0,\ q>0. \tag{2}$$

a°. The integral is convergent at ∞, for x arc cot x/p is limited in $(0,\infty)$; and at q, as a principal value,* since b° shows that Θ'_x is continuous near $x=q$.

b°. The integrand, $f(x,p)$, is a continuous function in any $R=(0,\infty;\ p_0<p_1)$; *i. e.*, for any $p_0>0$, except along $x-q=0$.

$f'_p(x,p)=\frac{x^2}{x^2-q^2}\cdot\frac{1}{p^2+x^2}$, $(p>0)$, which is continuous in R except along $x-q=0$.

Let $\frac{x}{x+q}\operatorname{arc\,cot}\frac{x}{p}=\Theta(x)$; $p>0$.

$$\Theta'_x=\frac{-px}{(x+q)(p^2+x^2)}+\frac{2x+q}{(x+q)^2}\cdot\operatorname{arc\,cot}\frac{x}{p};\quad p>0.$$

$$\Theta'_p=\frac{x^2p}{(x+q)(p^2+x^2)};\quad p>0.$$

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 24.

c°. We may differentiate under the integral sign in $R_1 = (0, q/2; p_0 < p_1)$. For both f and f'_p are continuous in R_1.

$$\frac{d}{dp}\int_0^{q/2} f\,dx = \int_0^{q/2} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

d°. In $R_2 = (q/2, 3q/2; p_0, p_1)$ we may differentiate* under the sign of integration (the principal value). For q is independent of p; and by b°, Θ'_x and Θ'_p are continuous in R_2.

$$\frac{d}{dp} P_q \int_{q/2}^{3q/2} f\,dx = P_q \int_{q/2}^{3q/2} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

e°. In $R_3 = (3q/2, \infty; p_0, p_1)$ we may do likewise.

For, $f(x, p)$ is regular in R_3 and continuous as to p for each x in $\mathfrak{A}_3$; $f'_p(x, p)$ is regular in R_3 and uniformly continuous in any $R'_3 = (3q/2 < b; p_0 < p_1)$, *i. e.*, for any $b > 3q/2$; $\int_{3q/2}^{\infty} f'_p\,dx$ converges uniformly in $\mathfrak{B} = (p_0 < p_1)$, since $|f'_p| \leqq \frac{x^2}{x^2 - q^2} \cdot \frac{1}{p_0^2 + x^2}$ for all p in $(p_0 < p_1)$, and this is integrable in $\mathfrak{A}_3 = (3q/2, \infty)$. So the conditions of Lect. I, p. 493, § 683, 2 are all satisfied and we have

$$\frac{d}{dp}\int_{3q/2}^{\infty} f\,dx = \int_{3q/2}^{\infty} \frac{\partial f}{\partial p}\,dx \quad \text{in } (p_0 < p_1).$$

f°. Combining c°, d°, e°, we have

$$\frac{dI}{dp} = P_q \int_0^{\infty} \frac{\partial f}{\partial p}\,dx, \quad \text{all } p \text{ in } (p_0 < p_1),\ p_0 > 0;\ \textit{i. e.}, \text{ for any } p > 0,$$

$$= -P_q \int_0^{\infty} \frac{x^2}{q^2 - x^2} \cdot \frac{1}{p^2 + x^2}\,dx, \quad \text{breaking up into partial fractions and using T 17, F 1, § 4,}$$

$$= \frac{p\pi}{2} \cdot \frac{1}{p^2 + q^2}; \quad q > 0,\ p > 0.$$

Integrating with respect to p from 0 to p (assigning to $\frac{dI}{dp}$ *any* value at $p = 0$), we have

$$I = \frac{\pi}{2}\int_0^{p} \frac{p\,dp}{p^2 + q^2}\,dx = \frac{\pi}{4}\log\frac{p^2 + q^2}{q^2}. \tag{2}$$

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXV, p. 84.

Criticism.

This proof follows the method of de Haan in VIII, p. 355, Méth. **10**, N **10**. De Haan's proof is formalistic, *e. g.*, the question of uniform convergence is not considered at all.

$$\int_0^1 \left(x \operatorname{arc}\cot x - \frac{1}{x} \operatorname{arc}\tan x\right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \log 2, \tag{3}$$

Nouv. Tab., T 232, F 1.

We shall establish the correctness of this evaluation.

a°. $P_1 \int_0^\infty \frac{x}{x^2-1} \operatorname{arc}\cot x \,.\, dx$

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} + \int_{1+\theta}^\infty \right\}$$

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} dx + \int_{1/1+\theta}^{\theta} dy \frac{-1}{y^2} \cdot \frac{1/y}{\frac{1}{y^2}-1} \,.\, \operatorname{arc}\tan y \right\};$$

changing variable,

$$= \lim_{\theta=0} \left\{ \int_0^{1-\theta} \frac{x \operatorname{arc}\cot x \, dx}{x^2-1} + \int_0^{1-\theta} \frac{1}{x} \cdot \frac{\operatorname{arc}\tan x}{x^2-1} \right\}$$
$$+ \lim_{\theta=0} \int_{1-\theta}^{1/1+\theta} \frac{1}{x} \cdot \frac{\operatorname{arc}\tan x \, dx}{x^2-1},$$

since by b° this last limit exists.

b°. $\lim_{\theta=0} \int_{1-\theta}^{1/1+\theta} \frac{1}{x} \cdot \frac{\operatorname{arc}\tan x \, dx}{x^2-1} = 0.$

For, $\left| \int_{1-\theta}^{1/1+\theta} \right| = \operatorname{arc}\tan \eta \int_{1-\theta}^{1/1+\theta} \frac{1}{x \,.\, (1-x^2)}$ (by the first Theorem of the Mean),

where $1-\theta \leqq \eta \leqq 1/1+\theta$,

$$\leqq \operatorname{arc}\tan \eta \left\{ \frac{1}{1+\theta} - (1-\theta) \right\} \frac{1}{(1-\theta) \,.\, \left(1 - \frac{1}{1+\theta}\right)}$$

$$= \operatorname{arc}\tan \eta \,.\, \frac{\theta^2}{1+\theta} \cdot \frac{1+\theta}{(1-\theta) \,.\, \theta}$$

$$= \operatorname{arc}\tan \eta \,.\, \frac{\theta}{1-\theta},$$

and therefore $\lim_{\theta=0} = 0.$

c°. Accordingly,

$$P_1 \int_0^\infty \frac{\operatorname{arc\,cot} x \,.\, x\, dx}{1-x^2} = \lim_{\theta=0} \int_0^{1-\theta} \left\{ \frac{x \operatorname{arc\,cot} x - \frac{1}{x} \operatorname{arc\,tan} x}{1-x^2} \right\} dx.$$

Using (2) $=$ T 248, F 10, with $p = q = 1$ ($=$ T 232, F 1), we have

$$-\frac{\pi}{4} \log 2 = \int_0^1 \frac{(x \operatorname{arc\,cot} x - \frac{1}{x} \operatorname{arc\,tan} x)\, dx}{1-x^2}, \tag{3}$$

and we do not need to inquire whether this integral is proper or improper.

Criticism.

In VIII, Méth. 10, N 10, p. 355, de Haan uses $P_1 \int_0^\infty \frac{x \operatorname{arc\,cot} x\, dx}{x^2-1} = \frac{\pi}{4} \log 2$ as the basis of the derivation. He immediately breaks this integral up into two divergent integrals $\int_0^1$ and $\int_1^\infty$ and then changes the variable in the latter of these. The proof here given avoids the use of divergent integrals.

Other formulas at once derived from (2) are:

$$P_q \int_0^\infty \operatorname{arc\,cot} \frac{x}{p} \,.\, \frac{x\, dx}{x^4 - q^4} = \frac{\pi}{8 q^2} \log \frac{p^2+q^2}{(p+q)^2}; \tag{4}$$

$q,\ p+q \neq 0$ ($=$ Nouv. Tab., T. 248, F 13).

$$P_q \int_0^\infty \operatorname{arc\,cot} \frac{x}{p} \,.\, \frac{x^3\, dx}{x^4 - q^4} = \frac{\pi}{8} \log \frac{(p+q)^2 (p^2+q^2)}{q^4}; \tag{5}$$

$p,\ q > 0$ ($=$ Nouv. Tab., T 248, F 14).

$$P_{\pi/4} \int_0^{\pi/2} \frac{\operatorname{arc\,tan}(p \tan x)\, dx}{\tan x \,.\, \cos 2x} = \frac{\pi}{4} \log (1+p^2); \tag{6}$$

$p > 0$ ($=$ Nouv. Tab., T 342, F 8).

In this last case make change of variable $x = \cot y$ and set $q = 1$. Then use Hardy's Theorem on Change of Variable.*

§ 8.

$$\int_{-\infty}^\infty \frac{\operatorname{arc\,tan}(a+bx)\, dx}{1+x^2} = \frac{\pi}{2} \left\{ \operatorname{arc\,tan} \frac{2a}{1-a^2-b^2} - \operatorname{arc\,tan} \frac{2ab}{1+a^2-b^2} \right\}; \tag{1}$$

$a > 0$, Nouv. Tab., T 254, F 10.

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* G. H. Hardy: *Proc. Lond. Math. Soc.*, Vol. XXXIV, p. 33.

We shall follow the outline of de Haan's method,* making his proof rigorous.

a°. We may differentiate with respect to b under the integral sign.

$f(x, b) = \dfrac{\arctan(a + bx)}{1+x^2}$ is regular in $R = (0, \infty;\ b_0 < b_1)$, $b_0 > 0$, and continuous in b for each x in $\mathfrak{A}$.

$f_b'(x, b) = \dfrac{x}{1+x^2} \cdot \dfrac{1}{1+(a+bx)^2}$ is regular in R; and uniformly continuous in b in $R' = (0, G;\ b_0, b_1)$ for any G, since it is continuous in R'.

$\int_0^\infty f_b'\,dx$ is uniformly convergent in $\mathfrak{B} = (b_0, b_1)$, since we have $|f_b'| \leqq \dfrac{x}{1+x^2} \cdot \dfrac{1}{1+(a+b_0x)^2}$ for all b in (b_0, b_1) and this is integrable in $\mathfrak{A}$.

Hence,† $\dfrac{d}{db}\int_0^\infty f\,dx = \int_0^\infty \dfrac{\partial f}{\partial b}\,dx$, in (b_0, b_1), $b_0 > 0$.

Similarly for $\int_{-\infty}^0$. Together: $\dfrac{d}{db}\int_{-\infty}^\infty f\,dx = \int_{-\infty}^\infty \dfrac{\partial f}{\partial b}\,dx$, for $b > 0$.

b°.
$$\begin{aligned}\frac{dI}{db} &= \int_{-\infty}^\infty \frac{\partial f}{\partial b}\,dx; \quad b > 0,\\ &= \int_{-\infty}^\infty \frac{x}{(1+x^2)\{1+(a+bx)\}^2}\,dx\\ &= \frac{1}{(1+a^2-b^2)^2+4a^2b^2}\Big[2ab\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx\\ &\qquad -(1+a^2)\,2a\int_{-\infty}^\infty \frac{b}{1+(a+bx)^2}\,dx\\ &\qquad +(1+a^2-b^2)\int_{-\infty}^\infty\left\{\frac{x}{1+x^2} - \frac{b^2x}{1+(a+bx)^2}\right\}dx\Big], \end{aligned} \tag{2}$$

where because the first two integrals converge, the third must; and because $a > 0$, $(1+a^2-b^2)^2 + 4a^2b^2 \neq 0$,

$$\begin{aligned}\frac{dI}{db} &= \frac{1}{(1+a^2-b^2)^2+4a^2b^2}\left[2ab\pi - (1+a^2)\,2a\pi + (1+a^2-b^2)\int_{-\infty}^\infty\{\ \}\right]\\ &= \frac{1}{(1+a^2-b^2)^2+4a^2b^2}\left[2a\pi(b-1-a^2) + (1+a^2-b^2)\int_{-\infty}^\infty\{\ \}\right]. \end{aligned} \tag{3}$$

* VIII, p. 355, Méth. 10, N 10.

† Lect. I, p. 493, § 683, 2.

c°. Evaluation of this last integral.

$\int_{-\infty}^{\infty}\left\{\frac{x}{1+x^2}-\frac{b^2 x}{1+(a+bx)^2}\right\}dx$ converges for $b>0$, and

$$= P_{\pm\infty}\int_{-\infty}^{\infty}\left\{\;\right\}, \text{ obviously,} \qquad (4)$$

$$= P_{\pm\infty}\int_{-\infty}^{\infty}\frac{x}{1+x^2}dx - P_{\pm\infty}\int_{-\infty}^{\infty}\frac{b^2 x\,dx}{1+(a+bx)^2}dx \qquad (5)$$

$$= 0 - P_{\pm\infty}\int_{-\infty}^{\infty}\frac{y-a}{y^2+a}dy, \text{ changing variable by } x=\frac{y-a}{b},\ b>0,$$

$$= -\lim_{\Theta=\infty}\int_{-\Theta}^{\Theta}\frac{d}{dy}\left[\tfrac{1}{2}\log(y^2+1)-a\arctan y\right]dy = a\pi. \qquad (6)$$

d°. Hence, $\frac{dI}{db}=\frac{-(1+a^2+b^2-2b)a}{(1+a^2-b^2)^2+4a^2b^2}$; $b>0$ [from (3) and (6)]. (7)

Give to $\frac{dI}{db}$ some value at the point $b=0$, and integrate with respect to b from 0 to b; we get finally:

$$I=\frac{\pi}{2}\left\{\arctan\frac{2a}{1-a^2+b^2}-\arctan\frac{2ab}{1+a^2-b^2}\right\}. \qquad (1)$$

Criticism.

(1) De Haan differentiates under the integral sign:

$$\frac{dI}{db}=\int_{-\infty}^{\infty}\frac{x}{1+(a+bx)^2}\cdot\frac{dx}{1+x^2}$$

$$=\frac{1}{(1+a^2-b^2)^2+4a^2b^2}\left[\int_{-\infty}^{\infty}\frac{2ab+(1+a^2-b^2)x}{1+x^2}dx - \int_{-\infty}^{\infty}\frac{(1+a^2)2ab+(1+a^2-b^2)b^2x}{1+(a+bx)^2}dx\right]. \qquad (8)$$

Now this step requires careful justification (see a°), and this is not given. The uniform convergence of $\int_{-\infty}^{\infty}f_b'\,dx$ must be considered. We find that $b=0$ must be excluded, for $\int_{-\infty}^{\infty}f_b'(x,0)\,dx$ does not converge, let alone converge uniformly.

(2) The integrals of the right member of (8) are divergent.

(3) Such conditions must be placed upon the constants that the denominator in (8) $= (1 + a^2 - b^2)^2 + 4a^2b^2 \neq 0$. A sufficient condition is $a > 0$.

(4) We note that the method of c°, where the principal value replaces the ordinary integral at (4), is a particularly useful one. The existence of the integral $\int_{-\infty}^{\infty}$ is known. Therefore also that of $P_{\pm\infty}$. Now, in the case of the former,

$$\int_{-\infty}^{\infty} \left\{ \frac{x}{1+x^2} - \frac{b^2 x}{1+(a+bx)^2} \right\} dx \neq \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx - b^2 \int_{-\infty}^{\infty} \frac{x}{1+(a+bx)^2} dx,$$

whereas

$$P\int_{-\infty}^{\infty} \left\{ \frac{x}{1+x^2} - \frac{b^2 x}{1+(a+bx)^2} \right\} dx = P\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx - b^2 P\int_{-\infty}^{\infty} \frac{x}{1+(a+bx)^2} dx,$$

and we at once lose the first of these, $P\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx = 0$, which means a great simplification.*

§ 9.

$$\int_a^b \arctan \frac{q}{x} \cdot \phi(x)\, dx = \int_0^{\infty} \frac{\sin qy\, dy}{y} \int_a^b e^{-xy} \phi(x)\, dx, \tag{1}$$

VIII, p. 452, Theorem XX.

We shall prove the following theorem:

Let $\phi(x)$ *be continuous and limited in* $\mathfrak{A} = (0, \infty)$.

Let $\phi(x)/x$ *be limited and integrable in* $(0^*, \infty)$.

$$\textit{Then } \int_0^{\infty} \arctan \frac{q}{x} \cdot \phi(x)\, dx = \int_0^{\infty} \frac{\sin qy\, dy}{y} \int_0^{\infty} e^{-xy} \phi(x)\, dx. \tag{2}$$

When $y = 0$, let the integrand $f = 0$, *i. e.*, $f(x, 0) = 0$.

In $R = (0, \infty;\ 0, \infty)$ $f(x, y)$ is continuous, except along $y = 0$; it is integrable in $\mathfrak{A}$, for every y in $\mathfrak{B}$; in $\mathfrak{B}$ for every x in $\mathfrak{A}$.

PART 1°. We shall consider $R_1 = (0, \infty;\ 0, 1)$ and show that

$$\int_0^{\infty} dx \int_0^1 f\, dy = \int_0^1 dy \int_0^{\infty} f\, dx.$$

Firstly, $f(x, y)$ is regular in R_1.

* Cf. G. H. Hardy: "On an Integral Equation," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. VII (1909), p. 450.

In the second place, $\int_0^\infty e^{-xy}\,\phi(x)\,dx$ converges uniformly in $\mathfrak{B}$, except on $y=0$, *i. e.*, in any $(\sigma<1)$; $\sigma>0$. For we have, for every y in $(\sigma<1)$, $|e^{-xy}\phi(x)|\leqq e^{-x\sigma}\,.\,M$, where M is some constant, since $\phi(x)$ is limited in $\mathfrak{A}$. And this is integrable in $\mathfrak{A}$.

Finally, we shall show $\int_0^\infty \phi(x)\,dx\int_0^y \frac{\sin qy}{y}\,e^{-xy}\,dy$ converges uniformly in (0, 1). This integral may be written:

$$\int_0^\infty \frac{\phi(x)\,dx}{x}\int_0^y \frac{\sin qy}{y}\,e^{-xy}\,x\,dy.$$

Now, by the First Theorem of the Mean,

$$\int_0^y \frac{\sin qy}{y}\,e^{-xy}\,x\,dy=\frac{\sin q\eta}{\eta}\int_0^y e^{-xy}\,x\,dy$$
$$=\frac{\sin q\eta}{\eta}\,(1-e^{-xy}),\quad \text{where } 0\leqq\eta\leqq y\leqq 1.$$

Accordingly, $\int_0^y \frac{\sin qy}{y}\,e^{-xy}\,x\,dy$ is limited in R_1, and for each y in $\mathfrak{B}$ is a monotone function of x, approaching 0 as x approaches ∞. Therefore, we have: $\frac{\phi(x)}{x}\int_0^y dy$ is regular in R_1, having no infinite discontinuities, being integrable in $\mathfrak{A}$ for each y in $\mathfrak{B}$ $\left(\frac{\phi(x)}{x}\text{ is integrable, }\int_0^y dy\text{ is monotone}\right)$, and integrable in $\mathfrak{B}$ for each x in $\mathfrak{A}$ $\left(\int_0^y dy\text{ is continuous}\right)$.

Now, since the integrand is regular, and breaks up into an integrable function, and one that is limited and monotone, we have* the uniform convergence of the above integral in the interval $\mathfrak{B}=(0,1)$.

We are now able† to state the result for Part 1°:

$$\int_0^\infty dx\int_0^1 f\,dy=\int_0^1 dy\int_0 f\,dx. \tag{3}$$

PART 2°. We now consider $R_2=(0,\infty\,;\,1,\infty)$ and show that

$$\int_0^\infty dx\int_1^\infty f\,dy=\int_1^\infty dy\int_0^\infty f\,dx.$$

(a) $f(x,y)$ is regular in R_2.

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* Lect. I, p. 468, § 661, 1. † Lect. I, p. 483, § 674, 1.

(b) $\int_0^\infty e^{-xy}\,\phi(x)\,dx$ is uniformly convergent in any $(1 < \beta)$, since $|e^{-xy}\,\phi(x)| \leqq e^{-x}\,.\,M$, $\phi(x)$ being limited in $\mathfrak{A}$.

(c) $\int_1^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy$ is uniformly convergent in any $(0 < b)$ except at $x = 0$, *i. e.*, in any $(s < b)$; $s > 0$. For, we have:

$$\left|\frac{\sin qy}{y}\,.\,e^{-xy}\right| \leqq q\,.\,e^{-sy} \text{ for all } x \text{ in } \mathfrak{A}.$$

(d) $\phi(x)\int_0^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy$ is integrable in any $(0 < b)$; for by the Second Theorem of the Mean, denoting the integrand by ψ,

$$\phi(x)\int_0^\infty \psi\,dy = \phi(x)\left\{e^{-x}\int_1^\eta \frac{\sin qy}{y}\,dy + 0\right\}$$
$$= \phi(x)\,.\,e^{-x}\,.\,(\text{a function of } y).$$

(e) $\int_0^\infty \phi(x)\,dx\int_1^y \frac{\sin qy}{y}\,e^{-xy}\,dy$ is uniformly convergent in $\mathfrak{B} = (1, \infty)$ since we have:

$$\left|\phi(x)\int_1^y \frac{\sin qy}{y}\,e^{-xy}\,dy\right| \quad \text{(by the Second Theorem of the Mean)}$$
$$= \left|\phi(x)\left\{e^{-x}\int_1 \frac{\sin qy}{y}\,dy + e^{-\ y}\int_\eta^y \frac{\sin qy}{y}\,dy\right\}\right|$$
$$\leqq |\phi(x)|\,.\,(e^{-x} + e^{-xy})\,.\,\max\left|\int_{y_1}^{y_2}\frac{\sin qy}{y}\,dy\right|$$
$$\leqq M\,.\,\pi\,.\,2e^{-x}, \quad \text{which is integrable in } (0, \infty).$$

Now (a), (b), (c), (d), (e) give us the conditions we require and we have:*

$$\int_0^\infty \phi(x)\,dx\int_1^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy = \int_1^\infty \frac{\sin qy\,dy}{y}\int_0^\infty e^{-xy}\,\phi(x)\,dx. \tag{4}$$

Now,

$$\int_0^\infty \frac{\sin qy}{y}\,e^{-xy}\,dy = \text{arc tan}\,\frac{q}{x}; \quad x > 0, \quad \text{Nouv. Tab., T 365, F 1,}\dagger$$

and when $x = 0$,

$$\int_0^\infty \frac{\sin qy}{y}\,dy, \text{ which } = +\pi/2,\ 0,\ -\pi/2 \text{ for } q > 0, = 0, < 0,$$
$$= \lim_{x=0} \text{arc tan}\,\frac{q}{x}. \tag{5}$$

(3), (4), (5) give (2).

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* Lect. I, p. 488, § 680, 1. † T 365, F 1, has been verified.

This theorem is stated by Cauchy.* De Haan † gives it as an immediate application of "Méthode 18" (cf. § 5). The conditions there are not sufficient, especially if b is infinite. In the application below b is infinite.

$$\int_0^\infty \operatorname{arc\,tan} \frac{q}{x} \cdot \sin px\, dx = \frac{\pi}{2p}(1 - e^{-pq}), \quad \text{Nouv. Tab., T 347, F 1.} \tag{6}$$

We shall add the conditions: $q > 0$, $p \neq 0$. (7)

An application of (2) above, ($\phi(x) = \sin px$) and Nouv. Tab., T 261, F 1.

We define $\operatorname{arc\,tan} \frac{q}{x} \sin px = \frac{\pi}{2}$, when $x = 0$.

We change $e^{-xy} \sin px$ to be 0, when $y = 0$, so that it may be integrable in $(0, \infty)$.

§ 10.

General Criticism.

1°. In the work of de Haan, no distinction is made between the principal value convergence of an integral over an infinite discontinuity and the unrestricted convergence in the case of our ordinary definition. One symbol is employed in both cases. De Haan's reason for always using the principal value definition is given in § 2, and concerns a single example, which is treated incorrectly.

The theorem, "If f is integrable in $\mathfrak{A} = (a < b)$, it is also integrable in $\mathfrak{B} = (\alpha < \beta)$, any partial field of $\mathfrak{A}$," holds good for principal value integrals *only* if α, β are not singularities across which f possesses only a principal value integral. In some instances (*e. g.*, see § 7) de Haan overlooks this point.

The notation of de Haan does not distinguish between the various kinds of principal value integrals. A list of some of these is given in § 2, 6°.

2°. Almost always the reader is without specific statements as to the range of application of the theorem; nearly all the formulas and theorems here considered are open to this criticism. This absence of conditions on functions and constants is very important.

It is tacitly assumed very often that certain letters may be infinite as well as finite. See §§ 5, 9.

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* "Œuv. Comp.", Sér. 2, Tome 1, p. 508.

† VIII, Méth. 18, N 14, p. 452, Theorem (XX).

Divergent integrals are sometimes introduced in the course of the reasoning, as *e. g.* in § 8.

3°. In double iterated limits, the order of passing to the limit is inverted without rigorous justification, practically always. The question of uniform convergence is never raised. See §§ 4, 5, 7, 8.

4°. The *methods* of de Haan, judged by modern standards, are usually incomplete or incorrect.

His *results* are remarkably free from error when one imposes proper limitations upon constants and functions. But as these limitations are not given, it is quite impossible to know, as far as the tables go, whether a formula is valid in a given case or not. Moreover, some results (see *e. g.* §§ 4, 6) are incorrect for *any* value of the constants.

NEW HAVEN, *August* 29, 1910.